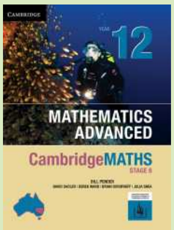
NORMANHURST BOYS HIGH SCHOOL

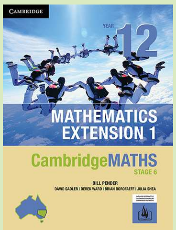
MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 12 COURSE

Topic summary and exercises:

A X1 X2 **Further Differentiation**

With references to





Name: ....

Initial version by H. Lam, November 2014 (Applications of Differentiation). Last updated March 25, 2024.
Various corrections by students & members of the Department of Mathematics at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under CC BY 2.0.

Symbols used

- Beware! Heed warning.
- Provided on NESA Reference Sheet
- Facts/formulae to memorise.
- Mathematics Extension 1 content.
- Literacy: note new word/phrase.
- Further reading/exercises to enrich your understanding and application of this topic.
- Syllabus specified content
- Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA12-1 uses detailed algebraic and graphical techniques to critically construct, model and evaluate arguments in a range of familiar and unfamiliar contexts

MA12-6 applies appropriate differentiation methods to solve problems

Syllabus subtopics

MA-F2 Graphing Techniques

MA-C3 Applications of Differentiation

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Advanced* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019a) or *CambridgeMATHS Year 12 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019b) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Part I

Curve sketching without calculus

Section 1

Asymptotes

Learning Goal(s)

Knowledge
Transformation of curves

Skills
Connect algebraic transformations to/from their graphical equivalents

Understanding
Asymptotes, number of solutions to equations, dilations

By the end of this section am I able to:

19.1 Apply transformations to sketch functions of the form $y = kf\Big(a(x+b)\Big) + c$, where $f(x)$ is a polynomial, reciprocal, absolute value and a, b, c and k are constants.

19.2 Use graphical methods with supporting algebraic working to solve a variety of practical problems involving any of the functions within the scope of this syllabus, in both real-life and abstract contexts

1.1 Vertical

(X1) See Pender et al. (2019b, Ex 5C).

Example 1

[2022 Hornsby Girls HS Adv Trial Q30] Suppose $f(x) = 2x^2 - x - 1$ and $g(x) = \cos x$.

Find the vertical asymptotes of $\dfrac{1}{f\Big(g(x)\Big)}$ where $x \in [0, \pi]$.

1.2 Horizontal

Steps

Finding horizontal asymptotes

1. Divide numerator and denominator by highest power of x

2. Use $\lim_{x\to\pm\infty} \frac{1}{x} = 0$ or $\lim_{x\to\pm\infty} \frac{1}{x^2} = 0$ as appropriate

Example 2

Examine the behaviour of the following function as $x \to \pm\infty$, noting any horizontal asymptotes:

$$y = \frac{3 - 5x - 4x^2}{4 - 5x - 3x^2}$$

Example 3

[**2015 Ext 1 HSC Q5**] What are the asymptotes of

$$y = \frac{3x}{(x+1)(x+2)}$$

(A) $y = 0,\ x = -1,\ x = -2$

(B) $y = 0,\ x = 1,\ x = 2$

(C) $y = 3,\ x = -1,\ x = -2$

(D) $y = 3,\ x = 1,\ x = 2$

Example 4

[2022 Sydney Grammar Adv Trial Q27] (1 mark) The graph of the function $y = \dfrac{x^2+3x}{x^2+3}$ has a horizontal asymptote. Find the equation of this asymptote, clearly showing your working.

Steps

1. Manipulate the numerator make it 'look' like the denominator

2. Split the fraction at the appropriate + or − sign from the numerator:

Example 5

[2012 Ext 1 HSC Q13]

i. Find the horizontal asymptote of the graph $y = \dfrac{2x^2}{x^2+9}$. **1**

ii. Without the use of calculus, sketch the graph of $y = \dfrac{2x^2}{x^2+9}$, showing the asymptote found in part (i). **2**

Example 6

[2013 Ext 1 HSC Q11] Consider the function $f(x) = \dfrac{x}{4 - x^2}$.

i. Show that $f'(x) > 0$ for all x in the domain of $f(x)$. **2**

ii. Sketch the graph of $y = f(x)$, showing all asymptotes. **2**

1.2.1 Further questions

1. Find the horizontal asymptotes to $y = \dfrac{2^x + 1}{2^x - 1}$.

2. [**2009 Ext 1 HSC Q4**] Consider the function $f(x) = \dfrac{x^4 + 3x^2}{x^4 + 3}$.

i. Show that $f(x)$ is an even function. **1**

ii. What is the equation of the horizontal asymptote to the graph $y = f(x)$? **1**

3. [**2017 VCE Mathematical Methods NHT Paper 2**]
Which of the following are equations of the asymptotes to $f(x) = \dfrac{3x - 5}{2 - x}$?

(A) $x = 2,\ y = \frac{5}{3}$

(B) $x = 2,\ y = -3$

(C) $x = -2,\ y = 3$

(D) $x = -3,\ y = 2$

(E) $x = 2,\ y = 3$

Further exercises

Ⓐ **Ex 2B**
- All questions

Ⓐ **Ex 2C**
- All questions

(X1) **Ex 3B**
- All questions

(X1) **Ex 3C**
- All questions

Section 2

Number of solutions of equations

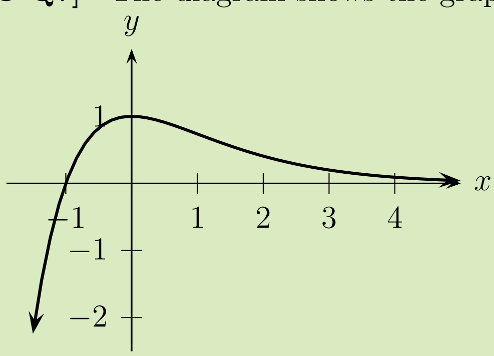
2.1 By curve sketching

Important note

- Questions of this nature requires a ......picture......
- They are *not* asking for ......where...... the solutions are!
- 'Un-simultaneous equation' expression.

Example 7

[2016 Fort Street HS 2U Q7] The diagram shows the graph of $y = e^{-x}(1+x)$.



How many solutions are there to the equation $e^{-x}(1+x) = 1 - \ln x$?

(A) 0 (B) 1 (C) 2 (D) 3

Example 8

[1997 3U HSC Q3]

i. On the same set of axes, sketch the graphs of $y = 2\sin\theta$ and $y = \theta$ for $-\pi \leq \theta \leq \pi$. **2**

ii. Use your sketch to find the number of solutions of the equation **1**

$$2\sin\theta = \theta \qquad \text{for } -\pi \leq \theta \leq \pi$$

Example 9

[2015 Sydney Grammar Ext 1 Trial Q8] How many solutions does the equation $x^{\frac{1}{3}} = |x-2| - 3$ have?

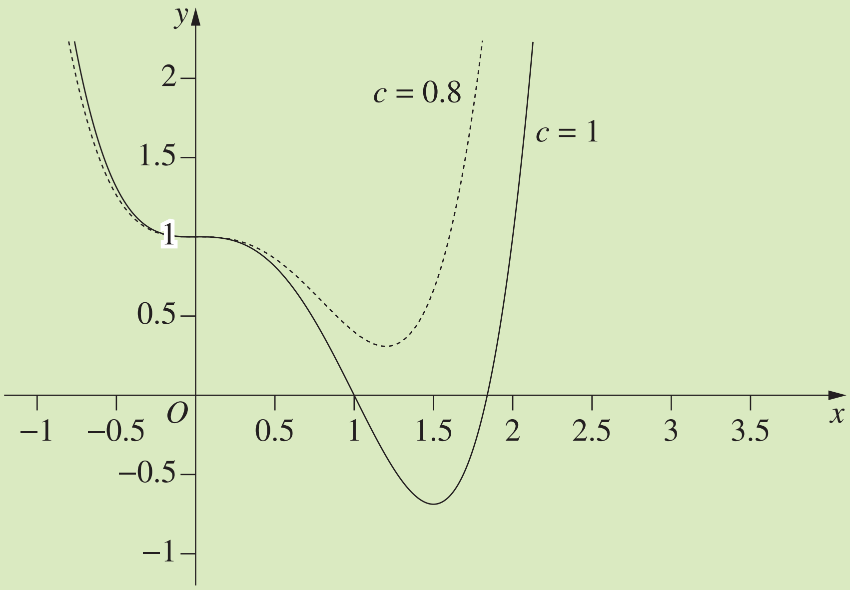
(A) 0 (B) 1 (C) 2 (D) 3

Example 10

[2023 Ext 1 HSC Q14] (X1) Consider the hyperbola $y = \dfrac{1}{x}$ and the circle $(x-c)^2 + y^2 = c^2$, where c is a constant.

i Show that the x coordinates of any points of intersection of the hyperbola and circle are zeros of the polynomial $P(x) = x^4 - 2cx^3 + 1$. **1**

ii The graphs of $y = x^4 - 2cx^3 + 1$ for $c = 0.8$ and $c = 1$ are shown. **3**



By considering the given graphs, or otherwise, find the exact value of $c > 0$ such that the hyperbola $y = \dfrac{1}{x}$ and the circle $(x-c)^2 + y^2 = c^2$ intersect at only one point.

Answer: $\frac{2}{\sqrt[4]{27}}$

2.2 By quadratic discriminant

Important note

Only use this when a quadratic equation can be formed. Otherwise, use the graphical techniques shown above as first point-of-call.

Example 11

The graphs of $y = mx + c$ and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

(A) $a > 0$ and $c > 0$

(B) $a > 0$ and $c < 0$

(C) $a > 0$ and $c > -\dfrac{m^2}{4a}$

(D) $a < 0$ and $c > -\dfrac{m^2}{4a}$

(E) $m > 0$ and $c < 0$

Answer: (D)

Further exercises

Most of this content has been covered in other topics. Only complete as much as required for the purposes of review.

Ⓐ **Ex 2E**

- All questions

(X1) **Ex 3E**

- All questions

Further questions

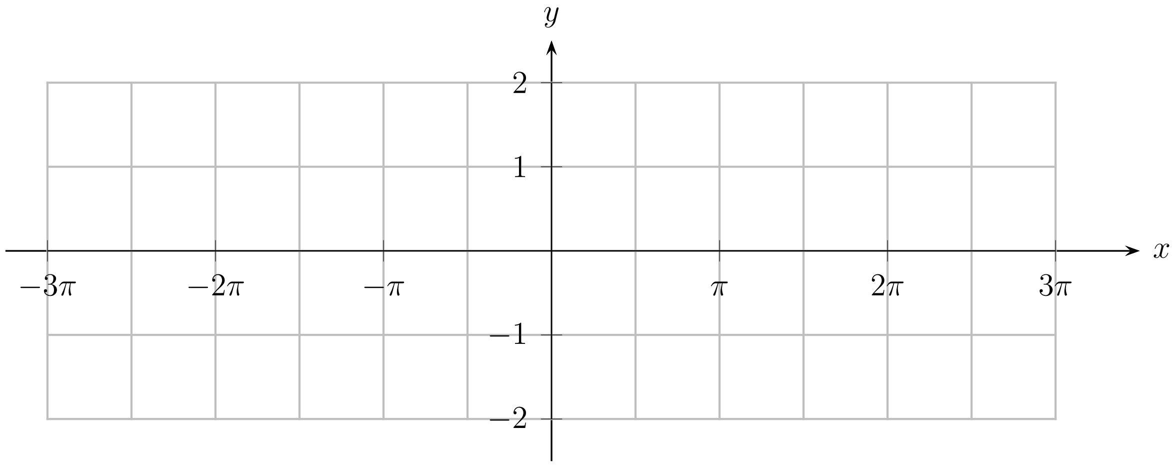
1. **[2015 Sydney Boys HS Ext 1 Trial Q12]**

i. On the same set of axes sketch the graphs of $y = \cos 2x$ and $y = \dfrac{x+1}{3}$. **2**

ii. Use the graph to determine the number of solutions to the equation **1**

$$3\cos 2x = x + 1$$

2. **[2020 Adv HSC Sample Q26]** By drawing graphs on the number plane, determine how many solutions there are to the equation $\sin x = \left|\dfrac{x}{5}\right|$ in the domain $(-\infty, \infty)$. **3**



Section 3

Translations and dilations

3.1 Review of Year 11 content

Theorem 1

If $a > 0$,

Translations

- $f(x-a)$ shifts right by a units.
- $f(x+a)$ shifts left by a units.
- $f(x)+a$ shifts up by a units.
- $f(x)-a$ shifts down by a units.

Reflections

- $-f(x)$: reflection about the x axis.
- $f(-x)$: reflection about the y axis.

Stretches

- $f(ax)$: horizontal stretch factor
- $af(x)$: vertical stretch factor

Further exercises

Most of this content has been covered in other topics. Only complete as much as required for the purposes of review.

Ⓐ **Ex 2F**

- All questions

X1 **Ex 3G**

- All questions

3.2 Dilations (Stretches)

Definition 1

Dilate to stretch out and make larger

- Vertical dilation factor of 2: $(x, y) \mapsto (x, 2y)$

  Description: hold x axis constant, then stretch vertically.

- Vertical dilation factor of $\frac{1}{3}$: $(x, y) \mapsto \left(x, \frac{1}{3}y\right)$

  Description: hold x axis constant, then compress vertically.

- Horizontal dilation factor of 2: $(x, y) \mapsto (2x, y)$

  Description: hold y axis constant, then stretch horizontally.

- Horizontal dilation factor of $\frac{1}{5}$: $(x, y) \mapsto \left(\frac{1}{5}x, y\right)$

  Description: hold y axis constant, then compress horizontally.

3.2.1 Horizontal dilations (stretches from y axis)

Steps

To stretch a graph in the horizontal direction by a factor of a from the y axis (*horizontal dilation*), replace x with $\frac{x}{a}$, i.e. new function rule is

$$y = f\left(\frac{x}{a}\right)$$

See Example 1 on page 22.

3.2.2 Vertical dilations (stretches from x axis)

Steps

To stretch a graph in the vertical direction by a factor of a from the x axis (*vertical dilation*), replace y with $\frac{y}{a}$, i.e. new function rule is

$$y = af(x)$$

Laws/Results

Combination of all transformations:

$$y = kf\Big(a(x+b)\Big) + c$$

1. a: ....horizontal........ stretch/compression (‘ ......dilation...... ’) ....factor....
   - $a \in (1, \infty)$: ....compression........
   - $a \in (0, 1)$: ......stretch......
2. b: ....horizontal........ shift (‘ ....translation........ ’) factor.
   - $b < 0$: shift ....right....
   - $b > 0$: shift ....left....
3. c: ......vertical...... shift (‘ ....translation........ ’) factor.
4. k: ......vertical...... stretch/compression (‘ ......dilation...... ’) ....factor....

Important note

Commence the transformation by working from the inner most part of the function notation outwards.

Example 12

[**2020 CSSA Adv Trial Q10**] The graph of $y = \dfrac{3}{x+1}$ is translated 4 units right and dilated vertically by a factor of $\dfrac{1}{2}$. Which of the following gives the equation of the new function?

(A) $\dfrac{y}{2} = \dfrac{3}{x-3}$ (B) $2y = \dfrac{3}{x-4}$ (C) $2y = \dfrac{3}{x-3}$ (D) $\dfrac{y}{2} = \dfrac{3}{x-4}$

Example 13

[2014 VCE Mathematical Methods (CAS) Paper 2 Q1] The point $P(4, -3)$ lies on the graph of a function f. The graph of f is translated four units vertically up and then reflected in the y axis.

What are the coordinates as a result of the transformations applied?
(A) $(-4, 1)$ (B) $(-4, 3)$ (C) $(0, -3)$ (D) $(4, -6)$ (E) $(-4, -1)$

Example 14

[2013 VCE Mathematical Methods (CAS) Paper 2 Q13] If the equation

$$f(2x) - 2f(x) = 0$$

is true for all real values of x, then the rule for f could be:

(A) $\dfrac{x^2}{2}$ (C) $2x$ (E) $x - 2$

(B) $\sqrt{2x}$ (D) $\log_e\left(\dfrac{|x|}{2}\right)$

Example 15

[2020 Sydney Girls HS Adv Trial Q10] The graph of $y = (x-2)^2(3+x)$ is dilated vertically by a factor of $\dfrac{1}{2}$ and horizontally by a factor of 3. It is then reflected across the y axis.

Which of the following is the new equation?

(A) $y = -2(3x-2)^2(3+3x)$

(B) $y = \dfrac{1}{2}(3x-2)^2(3+3x)$

(C) $y = -\dfrac{1}{2}\left(\dfrac{x}{3}-2\right)^2\left(3+\dfrac{x}{3}\right)$

(D) $y = \dfrac{1}{2}\left(\dfrac{x}{3}+2\right)^2\left(3-\dfrac{x}{3}\right)$

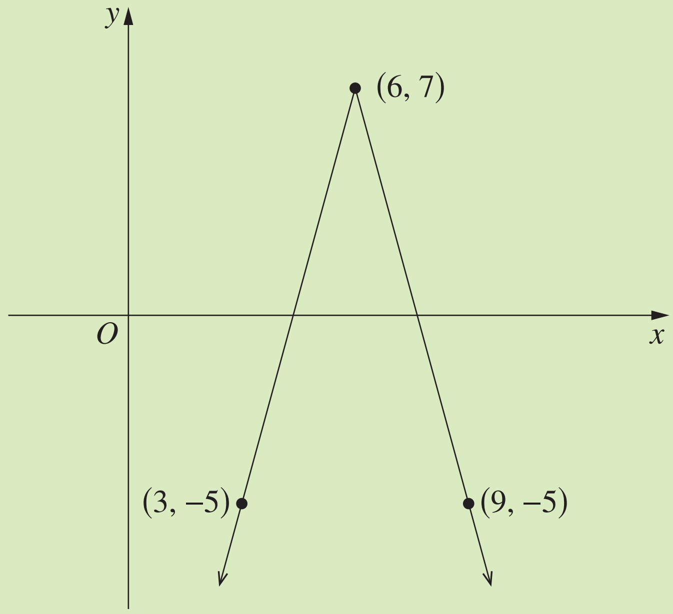
Example 16

[2022 Adv HSC Q19] (3 marks) The graph of the function $f(x) = x^2$ is translated m units to the right, dilated vertically by a scale factor of k and then translated 5 units down. The equation of the transformed function is $g(x) = 3x^2 - 12x + 7$.

Find the values of m and k. **Answer:** $m = 2$, $k = 3$

Example 17

[2023 Adv HSC Q27] The graph of $y = f(x)$, where $f(x) = a\,|x - b| + c$, passes through the points $(3, -5)$, $(6, 7)$ and $(9, -5)$ as shown in the diagram.



(a) Find the values of a, b and c. **3**

(b) The line $y = mx$ cuts the graph of $y = f(x)$ in two distince places. **2**

Find all possible values of m.

Further exercises

Ⓐ **Ex 2G-2I**
- All questions

(X1) **Ex 3G-3I**
- As many as required

Further questions

1. **[2015 VCE Mathematical Methods (CAS) Paper 2 Q11]** The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ on to the graph of $y = \sqrt{x^3 + 1}$ is a **1**

(A) dilation by a factor of 2 from the y axis

(B) dilation by a factor of 2 from the x axis

(C) dilation by a factor of $\frac{1}{2}$ from the x axis

(D) dilation by a factor of 8 from the y axis

(E) dilation by a factor of $\frac{1}{2}$ from the y axis

Answer: (A)

Part II

Applications of Differentiation

Section 4

Stationary points, second derivative and concavity

Learning Goal(s)

Knowledge
First derivative properties of a function

Skills
Analyse tables of values of first and second derivatives

Understanding
How to interpret the results of tables of values of first and second derivatives

By the end of this section am I able to:

19.3 Use the first derivative to investigate the shape of the graph of a function

19.4 Define and interpret the concept of the second derivative as the rate of change of the first derivative function in a variety of contexts, for example recognise acceleration as the second derivative of displacement with respect to time

Only complete as much as required for the purposes of reviewing the work from Year 11.

4.1 Increasing, decreasing, stationary

Further exercises

(A) **Ex 3A**
- Q7-23

(X1) **Ex 4A**
- Q6-12

4.2 Stationary points

Further exercises

(A) **Ex 3B**
- Q1-15

(X1) **Ex 4B**
- Q3-16

4.3 Second and higher derivatives

Further exercises

Ⓐ **Ex 3C**

- All questions

Ⓧ1 **Ex 4D**

- All questions

4.4 Concavity and points of inflexion

Now spelt *inflection* in the 2019 syllabuses.

Further exercises

Ⓐ **Ex 3D**

- Q3-19

Ⓧ1 **Ex 4E**

- Q4-22

Section 5

(X1) Critical values

Definition 2

A **critical value** occurs when

- $\frac{dy}{dx} = \underset{\dots}{0}$
- discontinuity in $\frac{dy}{dx}$

Example 18

(a) Find the critical values of $y = \frac{1}{x(x-4)}$, then use a table of test points of $\frac{dy}{dx}$ to analyse stationary points and find where the function is increasing and decreasing.

(b) Analyse the sign of the function in its domain, find any vertical and horizontal asymptotes, then sketch the curve.

Further exercises

(X1) **Ex 4C**

- Q1-11

Section 6

Curve Sketching

Learning Goal(s)

Knowledge
How to use differentiation tools to sketch graphs

Skills
Determine when to use a table of values or second derivative, and also product/quotient/chain rules

Understanding
When to avoid the table of values of first derivative to efficiently sketch graphs

By the end of this section am I able to:

19.5 Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

19.6 Use calculus to determine and verify the nature of stationary points, find local and global maxima and minima and points of inflection (horizontal or otherwise), examine behaviour of a function as $x \to \infty$ and $x \to -\infty$ and hence sketch the graph of the function

6.1 Curve sketching with calculus

Steps

Steps to sketching a curve $f(x)$ with calculus (abridged version of the "menu" of Pender et al. (2019b, p.157))

1. Find x and y intercepts, if possible.

2. Differentiate $f(x)$ to obtain $f'(x)$.

3. Find stationary points, i.e. solve ....$f'(x) = 0$.... .

4. *Determine the nature* of stationary points by either

- testing values of $f'(x)$ on the left and right of the values of x s.t. $f'(x) = 0$. (table)
- or, differentiate again to obtain $f''(x)$, and substitute values of x s.t. $f'(x) = 0$ into $f''(x)$.

5. Determine any points of inflexion (if requested).

6. Sketch the curve from the table.

Example 19

[2008 CSSA] Let $f(x) = 15 + 12x + 3x^2 - 2x^3$.

i Find the coordinates of the stationary points of $f(x)$ determine their nature. **3**

ii Find the coordinates of the point of inflexion. **1**

iii Sketch the graph of $y = f(x)$ indicating clearly the stationary points and point of inflexion. **2**

Example 20

[1997 NSG 3U Trial] (X1) Consider the graph $y = \dfrac{x^2}{1-x^2}$.

i. Write down the domain of this function. **1**

ii. Find the turning point and determine its nature. **3**

iii. Prove that the function is even. **1**

iv. Find $\displaystyle\lim_{x\to\infty} \frac{x^2}{1-x^2}$. **1**

v. Sketch the graph. **2**

Example 21

[1993 NSG 3U Trial] (X1) Consider the function $f(x) = \dfrac{x-2}{x^2+x-2}$.

i. For what values of x is $\dfrac{x-2}{x^2+x-2}$ undefined? **1**

ii. Where does the graph $f(x) = \dfrac{x-2}{x^2+x-2}$ cut the x axis? **1**

iii. Find the stationary points for the graph $f(x) = \dfrac{x-2}{x^2+x-2}$ and determine the nature by examining the first derivative only. **3**

iv. Sketch the graph of $f(x)$. **2**

Example 22

[2010 NSB Ext 1 Prelim Yearly] For the curve $y = x^5 - x^4$

i. Find the x intercepts of the curve. **1**

ii. Find and classify all stationary points of the curve. **3**

iii. Find the x coordinates of all points of inflexion. **3**

iv. Hence, sketch the graph, showing all important features. **2**

Further exercises

(A) **Ex 3E**
- All questions

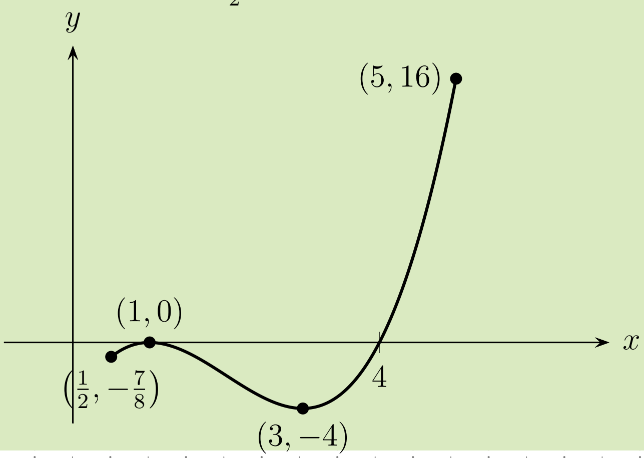
(X1) **Ex 4F**
- Q1-11

6.2 Global minimum/maximum

On a curve $y = f(x)$, a **global (absolute)*** minimum/maximum *may* occur, especially if a function has a .... restricted .... domain ....

Example 23

Identify the local minima/maxima and global minimum/maximum of the following curve, $y = x^3 - 6x^2 + 9x - 4$ where $\frac{1}{2} \leq x \leq 5$.



Example 24

Find the global maximum and minimum of $y = \dfrac{x+1}{x^2+3}$ for $x \geq 0$.

Answer: global max – $\frac{1}{2}$

Further exercises

(A) Ex 3F
- Q1-4

(X1) Ex 4G
- Q1-4

*as opposed to 'local'

Section 7

Optimisation

Learning Goal(s)

Knowledge
How to use differentiation techniques to solve optimisation problems

Skills
Use various other techniques, such as geometry, to obtain expressions that can then be differentiated

Understanding
Limitations to optimisation techniques, especially around boundary values

By the end of this section am I able to:

19.7 Solve optimisation problems for any of the functions covered in the scope of this syllabus, in a wide variety of contexts including displacement, velocity, acceleration, area, volume, business, finance and growth and decay

These problems are more widely known as *optimisation* problems.

- What is the least amount of materials required to hold a certain volume?
- How can cost be minimised to deliver the maximum number of goods?
- How can time be minimised to travel from A to B over a combination of land/sea?

Use local minima/maxima to assist in solving these problems. Set up the problem *properly* with the appropriate pronumeral(s).

Important note

- Draw picture!
- Think physically!
- Show all working!

7.1 Simple problems

Example 25

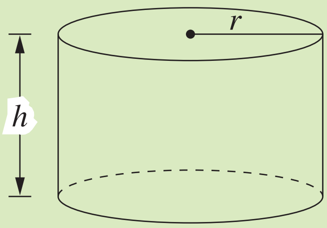
[Ex 10H Q8] A rectangle has a constant area of $36\,\text{cm}^2$.

(a) If the length of the rectangle is x, show that its perimeter is $P = 2x + \dfrac{72}{x}$.

(b) Show that $\dfrac{dP}{dx} = \dfrac{2(x-6)(x+6)}{x^2}$ and hence find the minimum possible perimeter.

Example 26

[2010 HSC Q5] A rainwater tank is to be designed in the shape of a cylinder with radius r metres and height h metres.



The volume of the tank is to be 10 cubic metres. Let A be the surface area of the tank, including its top and base, in square metres.

i. Given that $A = 2\pi r^2 + 2\pi rh$, show that $A = 2\pi r^2 + \dfrac{20}{r}$. **2**

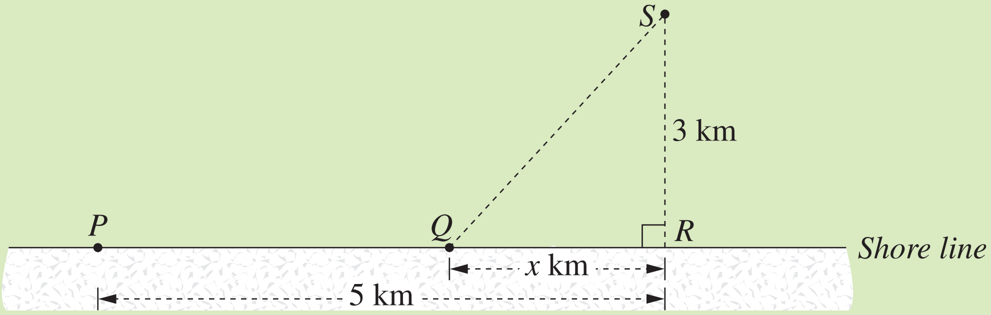
ii. Show that A has a minimum value and find the value of r for which the minimum occurs. **3**

Example 27

[**2009 HSC Q10**] An oil rig, S, is 3 km offshore. A power station, P, is on the shore. A cable is to be laid from P to S. It costs \$1 000 per kilometre to lay the cable along the shore and \$2 600 per kilometre to lay the cable underwater from the shore to S.

The point R is the point on the shore closest to S, and the distance PR is 5 km.

The point Q is on the shore, at a distance of x km from R, as shown in the diagram.



i. Find the total cost of laying the cable in a straight line from P to R and then in a straight line from R to S. **1**

ii. Find the cost of laying the cable in a straight line from P to S. **1**

iii. Let \$$C$ be the cost of laying the cable in a straight line from P to Q, and then in a straight line from Q to S. **2**

Show that $C = 1\,000\left(5 - x + 2.6\sqrt{x^2+9}\right)$.

iv. Find the minimum cost of laying the cable. **4**

v. New technology means that the cost of laying the cable underwater can be reduced to \$1 100 per kilometre. **2**

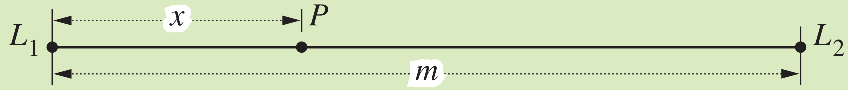
Determine the path for laying the cable in order to minimise the cost in this case.

Example 28

[2007 2U HSC Q10] The noise level, N, at a distance d metres from a single sound source of loudness L is given by the formula

$$N = \frac{L}{d^2}$$

Two sound sources, of loudness L_1 and L_2 are placed m metres apart.



The point P lies on the line between the sound sources and is x metres from the sound source with loudness L_1.

i. Write down a formula for the sum of the noise levels at P in terms of x. **1**

ii. There is a point on the line between the sound sources where the sum of the noise levels is a minimum. **4**

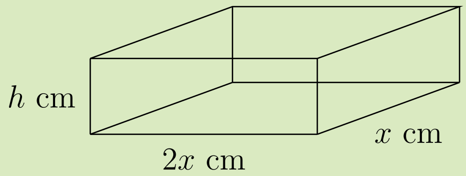
Find an expression for x in terms of m, L_1 and L_2 if P is chosen to be this point.

Example 29

[Ex 10G Q13/2011 Independent 2U] Joe is building a small *open topped* toy box. The box is twice as long as it is wide. The box has a total external surface area of $3\,750\,\text{cm}^2$.

Note: the box does not have a lid.

Answer: i. Show. ii. $25 \times 50 \times 16\frac{2}{3}$. iii. $19\,500\,\text{cm}^3$



i. Show that the height h of the toy box is given by $h = \dfrac{625}{x} - \dfrac{x}{3}$ **1**

ii. Find the dimensions of the box which give a maximum volume. **3**

iii. Joe decides that the height of the box cannot exceed $10\frac{5}{6}$ cm. **2**

Find the new dimensions of the box and hence find its volume if the surface area is to remain at $3\,750\,\text{cm}^2$.

Further exercises

(X1) Ex 4H

- Q1, 3, 5, 6
- Q14-22

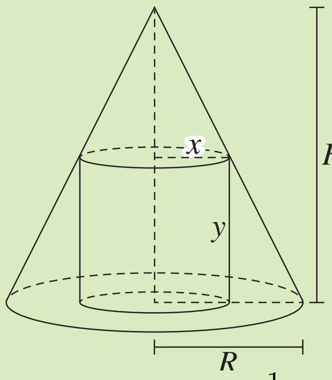
7.2 Optimisation with geometry

Important note

- Multidisciplinary problem! Use Euclidean/coordinate geometry, as well as quadratic methods to assist with setting up the question.
- Reduce to one independent variable only.

Example 30

[2015 HSC Q16/(X1) Ex 4I Q5] The diagram shows a cylinder of radius x and height y inscribed in a cone of radius R and height H, where R and H are constants.



The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

The volume of a cylinder of radius r and height h is $\pi r^2 h$.

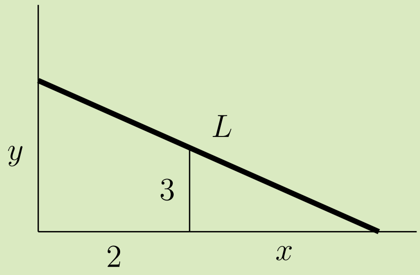
i. Show that the volume, V of the cylinder can be written as **3**

$$V = \frac{H}{R}\pi x^2(R - x)$$

ii. By considering the inscribed cylinder of maximum volume, show that the volume of any inscribed cylinder does not exceed $\frac{4}{9}$ of the volume of the cone. **4**

Example 31

[2011 NSB Ext 1 Prelim Yearly] ⚠ A 3 metre vertical fence stands 2 metres from a high vertical wall. A ladder is placed from the horizontal ground to the wall , resting on the fence. The base of the ladder is x metres from the fence.



i. If L represents the length of the ladder, show that **2**

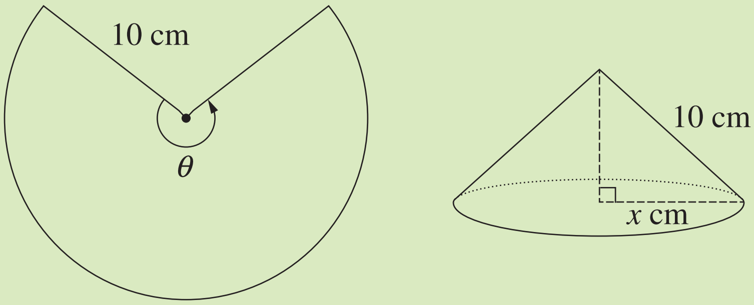
$$L^2 = (x+2)^2\left(1 + \frac{9}{x^2}\right)$$

ii. By first finding $\dfrac{dL^2}{dx}$, or otherwise, calculate the shortest ladder that can reach from the ground outside the fence to the wall, correct to 2 decimal places. **4**

Answer: $x = \sqrt[3]{18}$, $L \approx 7.02\,\text{m}$

Example 32

[**2018 2U HSC Q16**] A sector with radius 10 cm and angle θ is used to form the curved surface of a cone with base radius x cm, as shown in the diagram.



The volume of a cone of radius r and height h is given by $V = \dfrac{1}{3}\pi r^2 h$.

i. Show that the volume, V cm^3, of the cone described above is given by **1**

$$V = \frac{1}{3}\pi x^2\sqrt{100 - x^2}$$

ii. Show that $\dfrac{dV}{dx} = \dfrac{\pi x\left(200 - 3x^2\right)}{3\sqrt{100 - x^2}}$ **2**

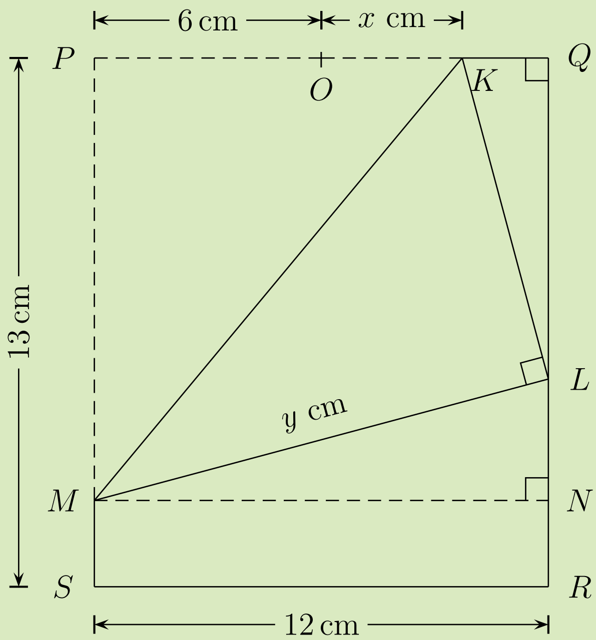
iii. Find the exact value of θ for which V is a maximum. **3**

Answer: i. Show ii. Show iii. $\frac{2\sqrt{6}\pi}{3}$

Example 33

[2006 2U HSC Q10] A rectangular piece of paper $PQRS$ has sides $PQ = 12\,\text{cm}$ and $PS = 13\,\text{cm}$. The point O is the midpoint of PQ. The points K and M are to be chosen on OQ and PS respectively, so that when the paper is folded along KM, the corner that was at P lands on the edge QR at L. Let $OK = x$ cm and $LM = y$ cm.

Answer: i. Show ii. Show iii. Show iv. $\frac{8}{3} \le x \le 6$ v. $\frac{169}{3}$



i. Show that $QL^2 = 24x$. **1**

ii. Let N be the point on QR for which MN is perpendicular to QR. **3**

By showing $\triangle QKL \parallel\!| \triangle NLM$, deduce that $y = \dfrac{\sqrt{6}\,(6+x)}{\sqrt{x}}$.

iii. Show that the area, A, of $\triangle KLM$ is given by $A = \dfrac{\sqrt{6}\,(6+x)^2}{2\sqrt{x}}$. **1**

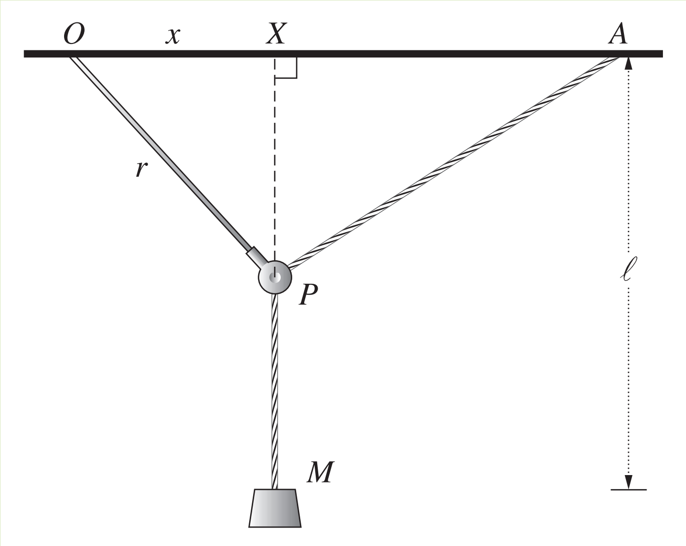
iv. Use the fact that $12 \le y \le 13$ to find the possible values of x. **2**

v. Find the minimum possible area of $\triangle KLM$. **3**

Example 34

[2003 2U HSC Q10] OMG! #Band6OrElse

A pulley P is attached to the ceiling at O by a piece of metal that can swing freely. One end of a rope is attached to the ceiling at A. The rope is passed through the pulley P and a weight is attached to the other end of the rope at M, as shown in the diagram.



The distance OA is 1 m, the length of the rope is 2 m, and the length of the piece of metal $OP = r$ metres, where $0 < r < 1$. Let X be the point where the line MP produced meets OA. Let $OX = x$ metres and $XM = \ell$ metres.

i. By considering $\triangle OXP$ and $\triangle AXP$, show that **1**

$$\ell = 2 + \sqrt{r^2 - x^2} - \sqrt{1 - 2x + r^2}$$

ii. Show that $\dfrac{d\ell}{dx} = \dfrac{(r^2 - x^2) - x^2(1 - 2x + r^2)}{\sqrt{r^2 - x^2}\sqrt{1 - 2x + r^2}\left(\sqrt{r^2 - x^2} + x\sqrt{1 - 2x + r^2}\right)}$. **2**

iii. You are given the factorisation **2**

$$\left(r^2 - x^2\right) - x^2\left(1 - 2x + r^2\right) = (x - 1)\left(2x^2 - r^2 x - r^2\right)$$

(Do NOT prove this.)

Find the value of x for which M is closest to the floor. Justify your answer.

Further exercises

(A) Ex 3G

- Q5-17

(X1) Ex 4I

- Q6-12

Section 8

Past examination questions

8.1 2022 Advanced HSC

Question 22(4 marks)

Find the global maximum and minimum values of $y = x^3 - 6x^2 + 8$, where $-1 \leq x \leq 7$. **4**

Question 27(7 marks)

Let $f(x) = xe^{-2x}$.

It is given that $f'(x) = e^{-2x} - 2xe^{-2x}$.

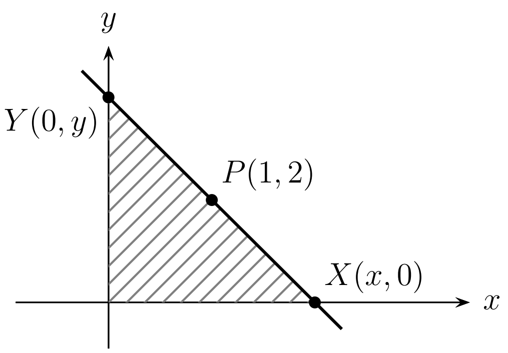
(a) Show that $f''(x) = 4(x-1)e^{-2x}$. **2**

(b) Find any stationary points of $f(x)$ and determine their nature. **2**

(c) Sketch the curve $y = xe^{-2x}$, showing any stationary points, points of inflection and intercepts with the axes. **3**

Question 31(6 marks)

A line passes through the point $P(1, 2)$ and meets the axes at $X(x, 0)$ and $Y(0, y)$, where $x > 1$.



(a) Show that $y = \dfrac{2x}{x-1}$ **2**

(b) Find the minimum value of the area of triangle XOY. **4**

8.2 2022 CSSA Trial

Question 26(3 marks)

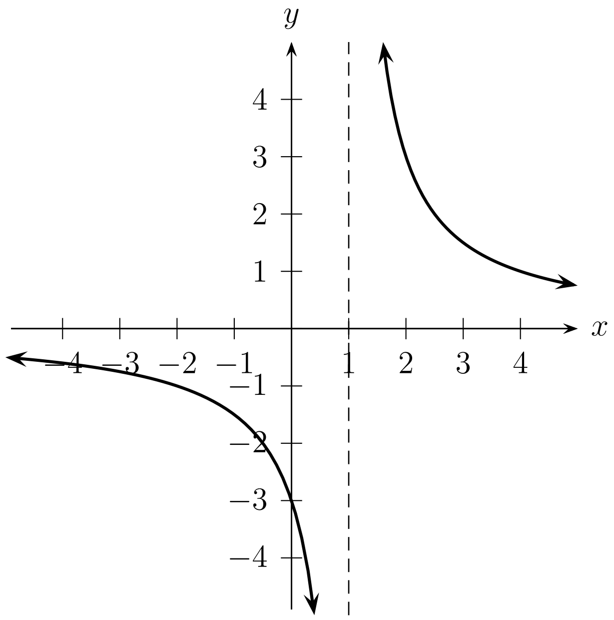
Let $f(x) = x^2 - 2x$.

Sketch the graph of $y = 2f(1 - x) - 6$, showing the location of the vertex.

8.3 2022 Sydney Grammar Trial

Question 18(3 marks)

The function $y = \dfrac{3}{x - 1}$ has been graphed on the coordinate plane below.



(a) Sketch the graph of $y = |x| - 2$ on the coordinate plane above, clearly showing any intercepts with the axes. **2**

(b) Hence, determine the number of solutions of the equation $|x| - \dfrac{3}{x - 1} = 2$. **1**

Question 27(1 mark)

The graph of the function $y = \dfrac{x^2 + 3x}{x^2 + 3}$ has a horizontal asymptote.

Find the equation of this asymptote, clearly show your working.

Question 28(2 marks)

(c) The graph of $y = g(x)$ is obtained by reflecting the graph of $y = x^3 + x^2 - 8x - 3$ in the x axis, shifting 5 units up and then dilating horizontally by a factor of $\frac{1}{2}$. **2**

Find the coordinates of the point A' being the image of the point A, after successive transformations have been applied to the point $A(-2, 9)$.

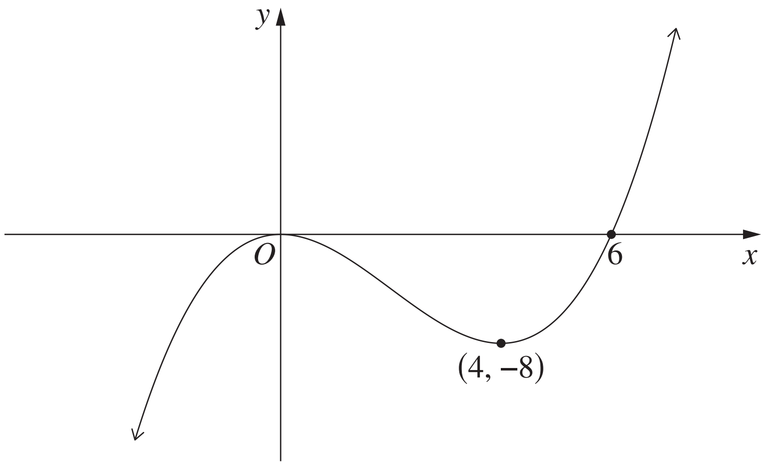
8.4 2021 Advanced HSC

Question 19(3 marks)

Without using calculus, sketch the graph of $y = 2 + \dfrac{1}{x+4}$, showing the asymptotes and the x and y intercepts.

Question 21(4 marks)

Consider the graph of $y = f(x)$ as shown.



Sketch the graph of $y = 4f(2x)$ showing the x intercepts and the coordinates of the turning points.

Question 31(4 marks)

By considering the equation of the tangent to $y = x^2 - 1$ at the point $(a, a^2 - 1)$, find the equations of the two tangents to $y = x^2 - 1$ which pass through $(3, -8)$.

Answer: $y = 14x - 50$, $y = -2x - 2$

8.5 2021 CSSA Trial

Question 29(4 marks)

Sketch the graph of the curve $y = x^3 + 3x^2 - 9x$, labelling the stationary points and the point of inflection. Do NOT determine the x intercepts of the curve.

8.6 2020 Advanced HSC

Question 16(4 marks)

Sketch the graph of the curve $y = -x^3 + 3x^2 - 1$, labelling the stationary points and point of inflection. Do NOT determine the x intercepts of the curve.

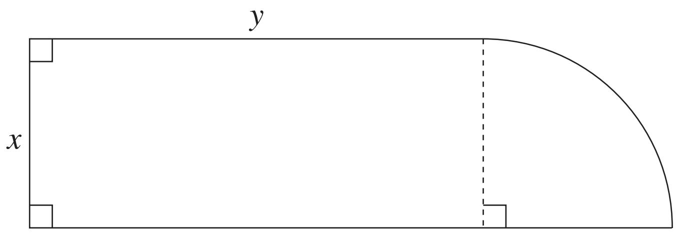
Question 24(3 marks)

The circle $x^2 - 6x + y^2 + 4y - 3 = 0$ is reflected in the x axis.

Sketch the reflected circle, showing the coordinates of the centre and radius.

Question 25(7 marks)

A landscape gardener wants to build a garden bed in the shape of a rectangle attached to a quarter-circle. Let x and y be the dimensions of the rectangle in metres, as shown in the diagram.



The garden bed is required to have an area of $36\,\text{m}^2$ and to have a perimeter which is as small as possible. Let P metres be the perimeter of the garden bed.

(a) Show that $P = 2x + \dfrac{72}{x}$ **3**

(b) Find the smallest possible perimeter of the garden bed, showing why this is the minimum perimeter. **3**

8.7 2020 CSSA Trial

Question 27(4 marks)

Consider the curve $y = (x+1)^2(x-5)$.

(a) Find the stationary points and determine their nature. **3**

(b) Given that the point $(1, -16)$ lies on the curve, show that it is a point of inflection. **2**

(c) Sketch the curve $y = (x+1)^2(x-5)$, showing the stationary points and the point of inflection. **2**

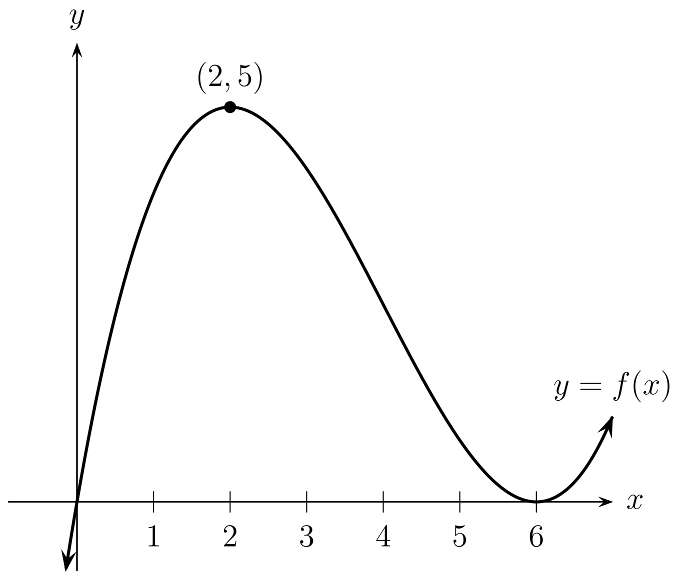
Question 28(3 marks)

Consider the circle given by the equation $x^2 + 8x + y^2 - 4y - 29 = 0$.

Give the equation of the circle in the form $(x-h)^2 + (y-k)^2 = r^2$, if it's translated up by three units and right by five units.

Question 31(2 marks)

The sketch shows $y = f(x)$.

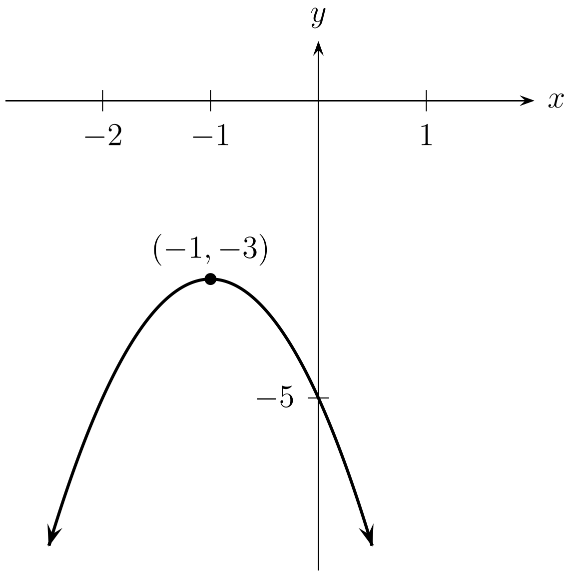


Draw a sketch of $y = f(2-x)$.

8.8 2020 Independent Trial

Question 30(3 marks)

The function $f(x) = x^2$ is transformed into a new functions whose graph is shown below.



Find the equation of the new function in the form

$$g(x) = kf(x + b) + c$$

for some constants k, b and c.

Question 31(3 marks)

(a) On the same diagram draw the graphs of $y = \cos \pi x$ and $y = 1 - |x|$ for $-3 \leq x \leq 3$. **2**

(b) Hence find the number of solutions to the equation $\cos \pi x = 1 - |x|$ in the domain $(-\infty, \infty)$. **1**

8.9 2019 2U HSC

Question 13

(e) i. Sketch the graph of $y = |x - 1|$ for $-4 \leq x \leq 4$. **1**

ii. Using the sketch from part (i), or otherwise, solve $|x - 1| = 2x + 4$. **2**

Question 14

(d) The equation of the tangent to the curve $y = x^3 + ax^2 + bx + 4$ at the point where $x = 2$ is $y = x - 4$. **3**

Find the values of a and b.

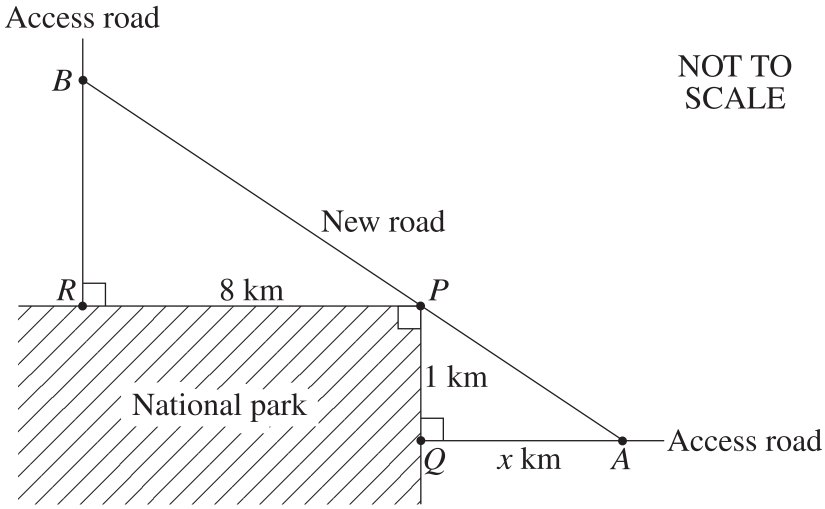
Question 15

(c) The entry points, R and Q, to a national park can be reached via two straight access roads. The access roads meet the national park boundaries at right angles. The corner, P, of the national park is 8 km from R and 1 km from Q. The boundaries of the national park form a right angle at P.

A new straight road is to be built joining these roads and passing through P.

Points A and B on the access roads are to be chosen to minimise the distance, D km, from A to B along the new road.

Let the distance QA be x km.



i. Show that $D^2 = (x+8)^2 + \left(\frac{8}{x}+1\right)^2$. **3**

ii. Show that $x = 2$ gives the minimum value of D^2. **3**

8.10 2017 2U HSC

Question 16

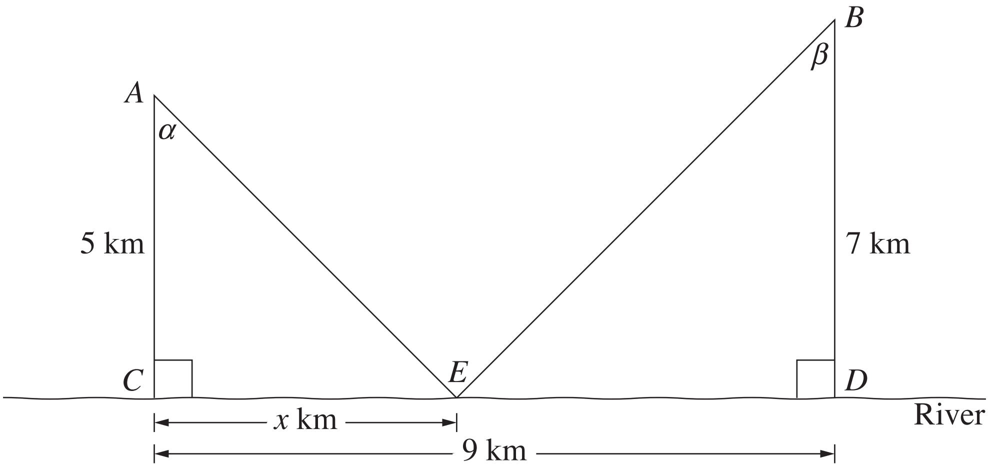
(a) John's home is at point A and his school is at point B. A straight river runs nearby.

The point on the river closest to A is point C, which is 5 km from A.

The point on the river closest to B is point D, which is 7 km from B.

The distance from C to D is 9 km.

To get some exercise, John cycles from home directly to point E on the river, x km from C, before cycling directly to school at B, as shown in the diagram.



The total distance John cycles from home to school is L km.

i. Show that $L = \sqrt{x^2 + 25} + \sqrt{49 + (9 - x)^2}$. **1**

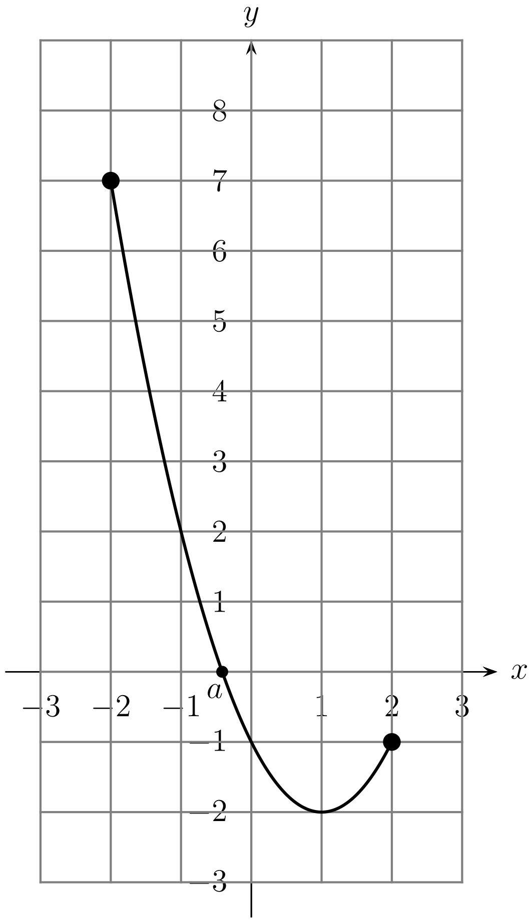
ii. Show that if $\dfrac{dL}{dx} = 0$, then $\sin\alpha = \sin\beta$. **3**

iii. Find the value of x that makes $\sin\alpha = \sin\beta$. **2**

iv. Explain why this value of x gives a minimum for L. **1**

8.11 2014 VCE Mathematical Methods (CAS) Paper 1

The graph of $f(x) = (x-1)^2 - 2,\ x \in [-2, 2]$ is shown. The graph intersects the x axis where $x = a$.



(a) Find the value of a. **1**

(b) (X1) On the same set of axes shown on the left, sketch the graph of $g(x) = |f(x)| + 1$ for $x \in [-2, 2]$. Label the end points with their coordinates. **1**

(c) The following sequence of transformations is applied to the graph of the function $g(x) = |f(x)| + 1$ such that $D_g = \{x : x \in [-2, 2]\}$.

- a translation of one unit in the negative direction of the x axis
- a translation of one unit in the negative direction of the y axis
- a dilation from the x axis of factor $\dfrac{1}{3}$.

i. The resultant algebraic rule $h(x)$ after the sequence of transformations has been applied to $g(x)$. **2**

ii. Find the domain of $h(x)$. **1**

Answer: (a) $a = 1 - \sqrt{2}$ (b) Sketch (c) i. $h(x) = \frac{1}{3}\left|x^2 - 2\right|$ ii. $x \in [-3, 1]$

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$l = \frac{\theta}{360} \times 2\pi r$

Area

$A = \frac{\theta}{360} \times \pi r^2$

$A = \frac{h}{2}(a + b)$

Surface area

$A = 2\pi r^2 + 2\pi rh$

$A = 4\pi r^2$

Volume

$V = \frac{1}{3}Ah$

$V = \frac{4}{3}\pi r^3$

Functions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For $ax^3 + bx^2 + cx + d = 0$:

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$(x - h)^2 + (y - k)^2 = r^2$

Financial Mathematics

$A = P(1 + r)^n$

Sequences and series

$T_n = a + (n - 1)d$

$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$

$T_n = ar^{n-1}$

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$

$S = \frac{a}{1 - r},\ |r| < 1$

Logarithmic and Exponential Functions

$\log_a a^x = x = a^{\log_a x}$

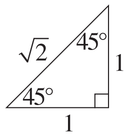
$\log_a x = \frac{\log_b x}{\log_b a}$

$a^x = e^{x \ln a}$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$



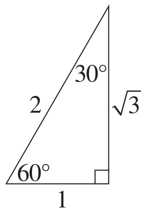
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \; \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \; \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \; \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

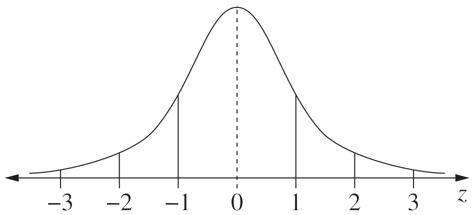
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \; P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \; P(X = x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \; x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

References

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019a). *CambridgeMATHS Stage 6 Mathematics Advanced Year 11* (1st ed.). Cambridge Education.

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